BAE 820 Physical Principles of Environmental Systems

Dimensionless analysis

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Dimensions and units

- Science depends on measured numbers, most of which have units. It is fairly easy to confuse the physical dimensions of a quantity with the units used to measure the dimension.
- We usually consider physical quantities like mass M, length L, time T, and temperature K, as fundamental dimensions. Other physical quantity often has a dimension which is a combination of multiple dimensions, which can be expressed as a product of powers of the basic dimensions M, L, T and K, e.g. speed, which is L/T.
- Units give the magnitude of some dimension relative to an arbitrary standard. In contrast to dimensions, of which only a few are needed, there is a multitude of units for measuring most quantities. Therefore, it is almost always necessary to attach a unit to the number. Without units, a number is at best meaningless and at worst misleading to the reader.

Dimensional consistency

- Dimensions and units must be handled consistently in any algebraic calculation. To be added, two quantities must have the same dimensions and units. Equations involving physical quantities must have the same dimensions and units on both sides.
- Physical laws are unaltered when changing the units measuring the dimensions.
- Verifying dimensional consistency (checking the units) is a very useful technique for uncovering errors in calculations.
- Handling dimensionless quantities should be easier, since they are pure numbers, with neither dimensions nor units.

Dimensional analysis

- Physical modeling often involves determining the relationship among variables that are in various dimensions. Considering the dimensions of those quantities can be useful when determining such relationship.
- Dimensional analysis is the analysis of the relationships between different physical quantities by identifying their fundamental dimensions (such as length, mass, and time) and units of measure (such as miles vs. kilometers, or pounds vs. kilograms vs. grams) and tracking these dimensions as calculations or comparisons are performed.
- It is generally used to categorize types of physical quantities and units based on their relationship to or dependence on other units. It can be used as a method for simplifying a mathematical model, since it can lead to a significant reduction of the number of variables.

Example of dimensional analysis

- The drag force F per unit length on a long smooth cylinder is a function of air speed U, density ρ, diameter D and viscosity μ. However, instead of having to draw hundreds of graphs portraying its variation with all combinations of these parameters, dimensional analysis tells us that the problem can be reduced to a single dimensionless relationship

$$c_d = f\ (R_e)$$

where c_d is the drag coefficient and Re is the Reynolds number.

- In this instance dimensional analysis has reduced the number of relevant variables from 5 to 2 and the experimental data to a single graph of c_d against R_e.

Dimensions of physical entities in the MLT system

Mass	M	Frequency	T^{-1}
Length	L	Momentum	MLT^{-1}
Time	T	Density	ML^{-3}
Velocity	LT^{-1}	Viscosity	$ML^{-1}T^{-1}$
Acceleration	LT^{-2}	Pressure	$ML^{-1}T^{-2}$
Force	MLT^{-2}	Surface tension	MT^{-2}
Energy, work	$ML^{2}T^{-2}$	Power	$ML^{2}T^{-3}$

The SI standard recommends the usage of the following dimensions and corresponding symbols: mass (M), length (L), time (T), electrical current (I), absolute temperature (Θ), amount of substance (N) and luminous intensity (J).

Example of dimensional analysis

- Find a model describing the terminal velocity of a particle that falls under gravity through a viscous fluid.
- We can assume that the velocity v depends on the particle's diameter D_p, the viscosity η and the acceleration g, and it is proportional to the difference between the density of the particle ρ_p and the density of the fluid ρ_l.
- Thus, we have

$$v=kD_p^a\eta^b g^c(\rho_p-\rho_l)$$

where k is the proportionality (dimensionless) constant and a; b and c are unknown numbers that we will determine using the dimensional analysis.

Example of dimensional analysis

- Consider the dimensions of both sides:

$$LT^{-1}=L^{a}(ML^{-1}T^{-1})^{b}(LT^{-2})^{c}(ML^{-3})$$

$$=M^{b+1}L^{a-b+c-3}T^{-b-2c}$$

- Equating the exponents of M, L and T on both sides we obtain a system of three equations in three unknowns.

$$0=b+1$$
$$1=a-b+c-3$$
$$-1=-b-2c$$

- The solution is a=2, b=-1, and c=1. Thus the equation for velocity becomes

$$v=\frac{kD_p^{2}(\rho_p-\rho_l)g}{\eta}$$

From dimensional analysis to dimensionless analysis

- In the above examples, we made several assumption, in a more general process, we need to consider dimensionless product of all the variables $v^a D_p{}^b g^c \rho^d \eta^e$.

$$M^0L^0T^0 = (LT^{-1})^a L^b (LT^{-2})^c (ML^{-3})^d (ML^{-1}T^{-1})^e$$

- This gives us a system of three equations and five unknowns.

$$d+e=0$$

$$a+b+c-3d-e=0$$

$$-a-2c-e=0$$

- Choose a and e as free variables, then $b=-\frac{1}{2}a-\frac{3}{2}e$, $c=-\frac{1}{2}a-\frac{1}{2}e$, $d=-e$.
- Thus, we obtain

$$v^a D_p{}^{-\frac{1}{2}a-\frac{3}{2}e} g^{-\frac{1}{2}a-\frac{1}{2}e} \rho^{-e} \eta^e = \left(\frac{v}{\sqrt{D_p g}}\right)^a \left(\frac{\eta}{\sqrt{D_p{}^3 g \rho^2}}\right)^e$$

- This gives us two dimensionless product : $\Pi_1 = \frac{v}{\sqrt{D_p g}}$, $\Pi_2 = \frac{\eta}{\sqrt{D_p{}^3 g \rho^2}}$.
- The modeling problem can be solved by developing a non-dimensional relationship between Π_1 and Π_2.

$$\Pi_1 = f(\Pi_2)$$

Dimensionless analysis

- **Buckingham's Pi Theorem**: If a problem involves n relevant variables, and m independent dimensions, then it can be reduced to a relation between a set of n-m non-dimensional groups (Πs).
- Mass transfer coefficient (K) is a function of the velocity (v), density (ρ), viscosity (η), and molecular diffusivity of the fluid (D), and some characteristics dimension of the system (L).

$$K = f(v, \rho, \eta, D, L)$$

- In the above system, number of non-dimensional groups is n-m = 6-3 = 3
- Dimensionless analysis provides us with equations relating the mass transfer coefficient to the properties of the system.

Important dimensionless numbers

- **The Reynold number**: Ratio of inertial forces to viscous forces

$$R_e = D\rho v/\eta = \frac{\rho v \pi D^3}{\eta \pi D^2} = \frac{inertial\ forces}{viscous\ forces}$$

- **The Schmidt number**: Ratio of momentum diffusivity (viscosity) and mass diffusivity

$$S_c = \eta/(\rho D) = \nu/D = \frac{viscous\ diffusion\ rate}{molecular\ (mass)\ diffusion\ rate}$$

- **The Sherwood number**: Ratio of convective to diffusive mass transport

$$S_h = KL/D = \frac{convective\ mass\ transfer\ coefficient}{diffusive\ mass\ transfer\ coefficient}$$

Where ν is kinetic viscosity, $\nu = \eta/\rho$

- In dimensionless analysis, the following dimensionless equation can be used to relate mass transfer coefficient to the properties of the system.

$$KL/D = f\,(D\rho v/\eta,\ \eta/\rho D)$$

$$S_h = f\,(R_e, S_c)$$

Let

$$S_h = \alpha R_e^{\beta} S_c^{\gamma}$$

The constants, α, β, and γ have to be determined experimentally.

The Reynold analogy

Mass transfer		Heat transfer	
Schmidt number	$S_c=\eta/\rho D=\nu/D$	Pranditl number	$P_r=C_p\eta/k=\nu/\alpha$
Sherwood number	$S_h=KL/D$	Nusselt number	$N_u=hL/k$

C_p is specific heat; k is thermal conductivity; α is thermal diffusivity; h is the convective heat transfer coefficient of the fluid.

- The following equation was developed for mass transfer in analogous to what has been found in heat transfer.

$$\frac{S_h}{R_e S_c{}^{1/3}} = \frac{f}{2}$$

Where f is the friction factor, which can be determined experimentally for various systems. f is often written as function of the Reynold number.

Example: Flat plate of length L

- When R_e<20,000 (laminar flow), f = 1.33/Re$^{1/2}$, Therefore

$$\frac{S_h}{R_e S_c^{1/3}} = \frac{f}{2} = \frac{1.33}{2R_e^{1/2}}$$

$$S_h = R_e S_c^{1/3} \frac{1.33}{2R_e^{1/2}} = 0.664 R_e^{1/2} S_c^{1/3}$$

$$K = \frac{D}{L} S_h = \frac{D}{L} 0.664 R_e^{1/2} S_c^{1/3}$$

- When R_e>20,000 (turbulent flow), f = 0.074/Re$^{0.2}$, Therefore

$$\frac{S_h}{R_e S_c^{1/3}} = \frac{f}{2} = \frac{0.074}{2R_e^{0.2}}$$

$$S_h = R_e S_c^{1/3} \frac{0.074}{2R_e^{0.2}} = 0.037 R_e^{0.8} S_c^{1/3}$$

$$K = \frac{D}{L} S_h = \frac{D}{L} 0.037 R_e^{0.8} S_c^{1/3}$$

Example: Bubble aeration

- For bubble aeration we have

$$S_h = 0.185 R_e S_c^{1/2} H_L^{-1/3}$$

- Where H_L is water depth in cm, it is included in the equation a correction.

$$a = \frac{A}{V} = \frac{V_{air}}{V_B}\times\frac{A_B}{V} = \frac{QH_L/UB}{\pi D_B{}^3/6}\times\frac{\pi D_B{}^2}{V} = \frac{6QH_L}{D_B U_B V}$$

Thus,

$$K_L a = \frac{D_L}{D_B}\times 0.185 R_e S_c^{1/2} H_L^{-1/3}\times\frac{6QHL}{D_B U_B V}$$
$$= \frac{D_L}{D_B}\times 0.185\times\frac{D_B U_B \rho}{\eta}\times S_c^{1/2} H_L^{-1/3}\times\frac{6QHL}{D_B U_B V}$$
$$= \frac{1.11 QHL^{2/3}}{S_c^{1/2}\, V D_B{}^2}$$

Where D_B is diameter of bubble, U_B is velocity of rise for bubbles, V is total volume of liquid, Q is air flow rate.

Problem solving #7

- We know that the speed of sound v in a gas depends on the pressure p and the density ρ. Find how the speed v depends on p and ρ.

Problem solution #7

- Pressure: $ML^{-1}T^{-2}$, Density: ML^{-3}, Velocity: LT^{-1}
- Let

$$v=kP^a\rho^b$$

$$LT^{-1}=(ML^{-1}T^{-2})^a\,(ML^{-3})^b$$

Thus

$$0=a+b$$

$$1=-a-3b$$

$$-1=-2a$$

- The solution is a=1/2, b=-1/2
- So we have

$$v = k\frac{\sqrt{p}}{\sqrt{\rho}}$$

where k is the proportionality constant.