BAE 820 Physical Principles of Environmental Systems

Stokes' law and Reynold number

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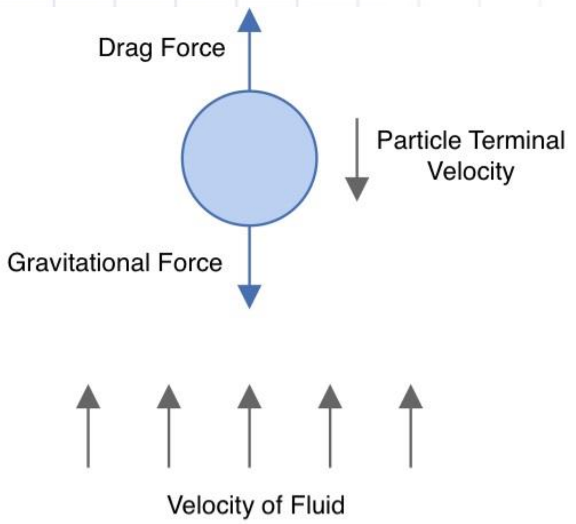
The motion of a particle in a fluid environment, such as air or water



$$m\frac{dV}{dt} = F(t) - F_d - \frac{1}{2}\frac{4}{3}\pi r^3 \rho \frac{dV}{dt} - \frac{4}{3}\pi r^3 \frac{dp}{dr} - F_B$$

Where,

- F(t) is external force.
- F_d is the resistance of the medium for a constant velocity.
- The third term represents the resistance to accelerated motion of the particle.
- The fourth term represents the resistance due to the pressure gradient across the particle.
- The fifth term is the Basset force (F_B), which represents the energy recovery expended in setting the medium itself in motion.

Newton's resistance law

$$F_d = \frac{1}{2} c_d \rho v^2 A$$

- F_d is the drag force of the fluid medium
- ρ is the mass density of the fluid
- v is the speed of the object relative to the fluid
- A is the projected area, for a sphere, $A = \pi r^2$
- c_d is the dimensionless drag coefficient, which is not a constant but varies as a function of speed, flow direction, object position, object size, fluid density and fluid viscosity.

Stoke's law

Under laminar flow conditions (undisturbed flow), the force of viscosity on a small sphere moving through a viscous fluid is given by

$$F_s = 6\pi r\eta v = \xi v$$

- F_s is the frictional force – known as Stokes' drag.
- η is the dynamic viscosity, kg /(m·s).
- r is the radius of the spherical object, m.
- v is the speed of the object relative to the fluid, m/s.
- ξ is the friction factor, kg·m/s^2, in which 2/3 is due to skin friction, and 1/3 is due to pressure forces.

Under laminar flow conditions, setting the two expressions equal to each other we get

$$6\pi r\eta v = \frac{1}{2} c_d \rho v^2 A$$

Therefore,

$$C_d = 12\,\eta/r\rho v = 24\eta/D\rho v$$

- D is the diameter of the spherical object

Define Reynold number,

$$R_e = D\rho v/\eta = Dv/\nu$$

- Where $\nu = \eta/\rho$, is called the kinematic viscosity, m^2/s.

So, the dimensionless drag coefficient

$$C_d = 24/R_e$$

The Reynold number (R_e)

- R_e is a dimensionless quantity that is used to characterize different flow regimes, such as laminar or turbulent flow.
- R_e can be defined as the ratio of inertial forces to viscous forces and quantifies the relative importance of these two types of forces for given flow conditions.

$$R_e = D\rho v/\eta = \frac{\rho v \pi D^3}{\eta \pi D^2} = \frac{inertial\ forces}{viscous\ forces}$$

- R_e can be used to determine dynamic similitude between two different cases of fluid flow. It frequently arise when performing scaling of fluid dynamics problems.

The Reynold number (R_e)

- R_e can be defined for several different situations where a fluid is in relative motion to a surface.

	Laminar flow	Turbulent flow
Fluid around sphere object	$R_e<1$	$R_e>700$
Fluid in pipe	$R_e<2,100$	$R_e>10,000$

Magnitude of the kinematic viscosity and R_e

Fluid	η (Pa s) at 20ºC	ν (m^2/s) at 20ºC	R_e (when D=1m and v=1m/s)
Air	1.8×10^{-5}	1.5×10^{-5}	$\sim7\times10^{4}$
Water	1×10^{-3}	1×10^{-6}	$\sim1\times10^{6}$

- Most surface water systems are turbulent.
- Groundwater flow is nearly always laminar since the characteristic length scale for groundwater systems is the pore scale.

R_e of spherical particles falling at their terminal velocities in air

Dp (μm)	R_e (at 298 K and 1 atm)
20	0.02
60	0.4
100	2
300	20

- Thus, for particles smaller than about 20 μm in diameter, Stokes' law is an accurate formula for the drag force exerted by the air.

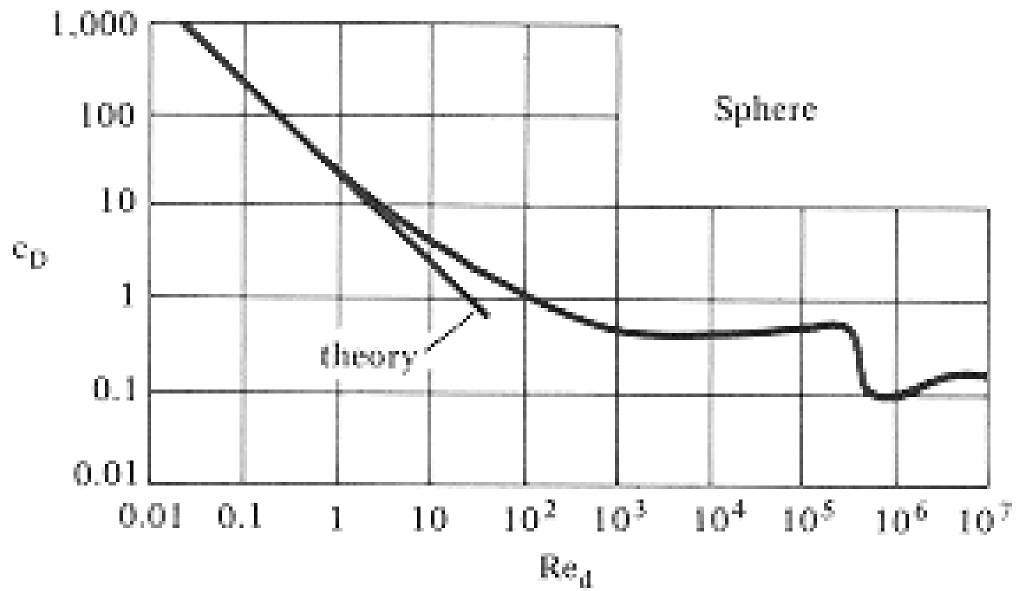
Relationship between R_e and c_d

- For $R_e<1$, $c_d=24/R_e$
- For $1 < R_e < 1{,}000$, transitional flow, $c_d=24/R_e+3/\sqrt{R_e}+0.34$

An approximate form is

$$c_d=b/R_e^a$$

- For $1{,}000 < R_e < 100{,}000$, c_d =constant ~ 0.44
- At R_e ~ 500,000, c_d drops by a factor of about five and a new flow situation has been established.



From: Wegener, "What Makes Airplanes Fly?" Springer-Verlag (1991)

The motion of a particle in a fluid environment

$$m\frac{dV}{dt}=F(t)-F_d$$

What is the settling velocity (v_t)?

- When Stokes law is obeyed,

$$6\pi r\eta v_t=F(t)=\frac{4}{3}\pi r^3(\rho_p-\rho_l)g \;\; => \;\; v_t=\frac{2r^2(\rho_p-\rho_l)g}{9\eta}$$

- When Stokes law is not obeyed,

$$\frac{1}{2}c_d\rho_l v_t^2 A=F(t)=\frac{4}{3}\pi r^3(\rho_p-\rho_l)g \;\; => \;\; v_t^2=\frac{8r(\rho_p/\rho_l-1)g}{3cd}$$

A trail and error procedure

1. Calculate v_t using turbulent or laminar flow equation based on assumptions from your best guess.
2. Plug v_t into $R_e = D\rho v_t/\eta$, check R_e. If your initial assumption is valid, then problem is solved.
3. Make new assumptions based on calculated R_e. Use R_e to calculate c_d.
4. Use c_d to calculate v_t.
5. If v_t is not equal to previous v_t, using the new value for v_t and repeat from step 2.

Problem Solving #1

- Calculate settling velocity (v_t) of a particle with diameter=10 μm and 100μm respectively, in air at 298 K and 1 atm. Assuming density of the particle ρ_p=1000kg/m^3

Problem Solution #1

ρ_p=1000kg/m³; ρ_l=1.20 kg/m³

- For D_p=10μm

Assume laminar condition,

$$v_t = \frac{2r^2(\rho_p - \rho_l)g}{9\eta} = \frac{2(10/2\times10^{-6})^2(1000-1.20)\times9.8}{9(1.8\times10^{-5})} = 3.02\times10^{-3}\text{m/s}$$

Check

$$R_e = D_p\rho_l v_t/\eta = \frac{(10\times10^{-6})(1.20)\times3.02\times10^{-3}}{(1.8\times10^{-5})} = 2.01\times10^{-3} << 1$$

So, the assumption of laminar condition is valid.

- For D_p=100μm

Assume laminar condition,

$$v_t = \frac{2r^2(\rho_p - \rho_l)g}{9\eta} = \frac{2(100/2\times10^{-6})^2(1000-1.20)\times9.8}{9(1.8\times10^{-5})} = 0.30\text{m/s}$$

Check

$$R_e = D_p\rho_l v_t/\eta = \frac{(100\times10^{-6})(1.20)\times0.3}{1.8\times10^{-5}} = 2>1$$

So, the assumption of laminar condition is not valid.

In transitional flow,

$c_d=24/R_e+3/\sqrt{R_e}+0.34 = 24/2+3/\sqrt{2}+0.34 = 14.5$

$$v_t^2 = \frac{8r(\rho_p/\rho_l-1)g}{3cd} = \frac{8\times100/2\times10^{-6}\times(1000/1.2-1)\times9.8}{3\times14.5} = 0.075$$

$v_t = 0.27$ m/s

Not same with the initial value of v_t, so continue the procedure,

$$R_e=D_p\rho_l v_t/\eta = \frac{(100\times10^{-6})(1.20)\times0.27}{1.8\times10^{-5}} = 1.8>1$$

$c_d=24/R_e+3/\sqrt{R_e}+0.34 = 24/1.8+3/\sqrt{1.8}+0.34 = 15.9$

$$v_t^2 = \frac{8r(\rho_p/\rho_l-1)g}{3cd} = \frac{8\times100/2\times10^{-6}\times(1000/1.2-1)\times9.8}{3\times15.9} = 0.068$$

$v_t = 0.26$ m/s

Not same with the previous value of v_t, so continue the procedure,

$$R_e = D_p \rho_l v_t / \eta = \frac{(100\times10^{-6})(1.20)\times0.26}{1.8\times10^{-5}} = 1.7 > 1$$

$$c_d = 24/R_e + 3/\sqrt{R_e} + 0.34 = 24/1.7 + 3/\sqrt{1.7} + 0.34 = 16.8$$

$$v_t^2 = \frac{8r(\rho_p/\rho_l - 1)g}{3cd} = \frac{8\times100/2\times10^{-6}\times(1000/1.2-1)\times9.8}{3\times16.8} = 0.065$$

$v_t = 0.25$ m/s

Not same with the previous value of v_t, so continue the procedure,

$$R_e = D_p \rho_l v_t / \eta = \frac{(100\times10^{-6})(1.20)\times0.25}{1.8\times10^{-5}} = 1.7 > 1$$

$$c_d = 24/R_e + 3/\sqrt{R_e} + 0.34 = 24/1.7 + 3/\sqrt{1.7} + 0.34 = 16.8$$

$$v_t^2 = \frac{8r(\rho_p/\rho_l - 1)g}{3cd} = \frac{8\times100/2\times10^{-6}\times(1000/1.2-1)\times9.8}{3\times16.8} = 0.065$$

$v_t = 0.25$ m/s

Same with the previous value of v_t. At this point, we may consider the problem solved. $v_t = 0.25$ m/s can be the final answer.